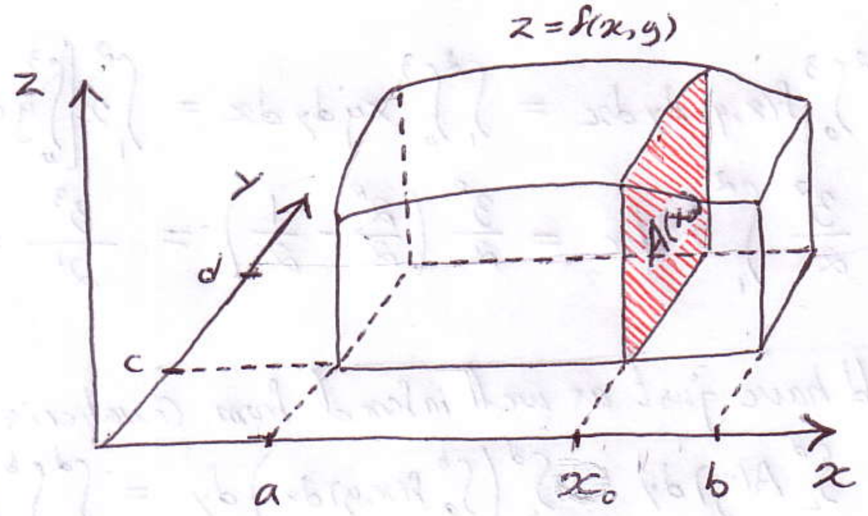
(5.2)

The double Integral

Suppose that Aunt Jemima bakes you a cake with rectangular base. Naturally, cakes are calorie bombs and the amount of calories in the entire cake is proportional to its volume.
How do you go about computing this volume?
Let $R = \{(x,y) : a \le x \le b,\ c \le y \le d\}$ be the rectangular base of the cake and let $z = f(x,y)$ be the cake's surface over R.
We are interested in the volume enclosed between the surface $z = f(x,y)$ and the rectangle R.



First, let us agree to denote the volume by $\iint_R f$ or $\iint_R f(x,y)\,dy\,dx$
(The reason for the double integral sign will become clear in just a moment).
If we can find, $A(x)$, the area of the cross section as a function of x, we will be able to use Cavalieri's principle to compute $\iint_R f$.
In particular, $\iint_R f = \int_a^b A(x)\,dx$.

If we fix $x = x_0$, $x_0 \in [a,b]$, y still varies from c to d. The height on the surface $z = f(x,y)$ is now only a function of y. Specifically, when $x = x_0$, the height above the surface is a function $g: [c,d] \to \mathbb{R}$, given by $g(y) = f(x_0, y)$. By single-variable calculus, the area $A(x_0)$ is just the area above the interval $[c,d]$ and below g. That is
$A(x_0) = \int_c^d g(y)\,dy = \int_c^d f(x_0, y)\,dy$

Therefore $\iint_R f = \int_a^b A(x)\,dx = \int_a^b \left(\int_c^d f(x,y)\,dy\right)dx. = \int_a^b \int_c^d f(x,y)\,dy\,dx$

Ex. Let $R = \{(x,y) : x \in [1,2],\ y \in [0,3]\}$ and let $f(x,y) = xy$

Compute $\iint_R f$.

Solution: $\iint_R f = \int_1^2 \int_0^3 f(x,y)\,dy\,dx = \int_1^2 \int_0^3 xy\,dy\,dx = \int_1^2 x\left[\int_0^3 y\,dy\right]dx =$
$= \int_1^2 x\left[\frac{y^2}{2}\right]_0^3 dx = \frac{3^2}{2}\int_1^2 x\,dx = \frac{3^2}{2}\left(\frac{2^2}{2} - \frac{1}{2}\right) = \frac{3^3}{2^2} = \frac{27}{4}$

Observe that we could have just as well inferred from Cavalieri's principle that $\iint_R f = \int_c^d A(y)\,dy = \int_c^d \left(\int_a^b f(x,y)\,dx\right)dy = \int_c^d \int_a^b f(x,y)\,dx\,dy$.

Ex. Compute $\iint_R (x^2+y^2)\,dx\,dy$ where $R = [-1,1] \times [0,1]$

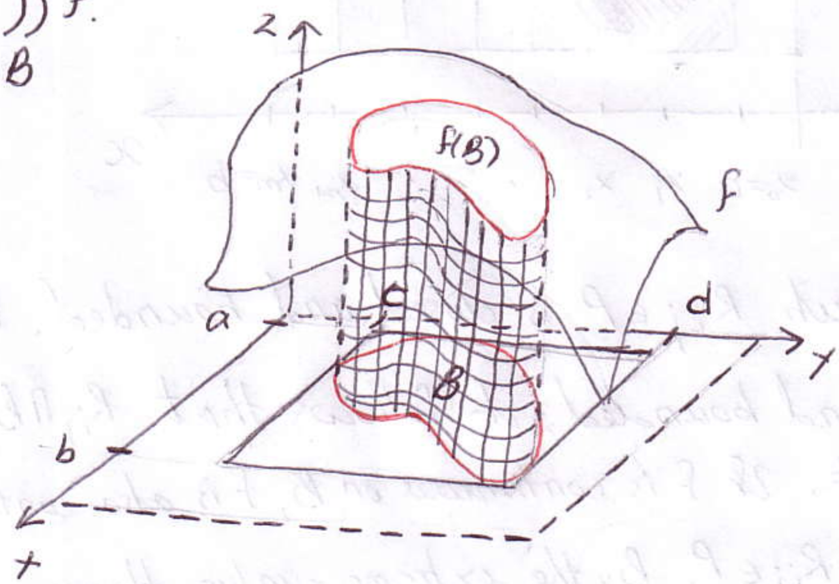
Solution: This double integral may be written as either
$\int_{-1}^1 \int_0^1 (x^2+y^2)\,dy\,dx$ or as $\int_0^1 \int_{-1}^1 (x^2+y^2)\,dx\,dy$.

Now $\int_0^1 \int_{-1}^1 (x^2+y^2)\,dx\,dy = \int_0^1 \left(\frac{2(1)^3}{3} + 2y^2\right)dy = \int_0^1 \left(\frac{2}{3} + 2y^2\right)dy =$
$= \frac{2}{3} + \frac{2}{3} = \frac{4}{3} = \int_{-1}^1 \int_0^1 (x^2+y^2)\,dy\,dx$ (What sort of surface is $z = x^2+y^2$?)

Double Integrals as Riemann sums

Although we are lacking necessary tools from analysis to define double integrals rigorously, we will need to understand a thing or two about Riemann sum representation of a double integral nonetheless.

Suppose B is a closed and bounded region in $\mathbb{R}^2$. We would like to define $\iint\limits_B f$.



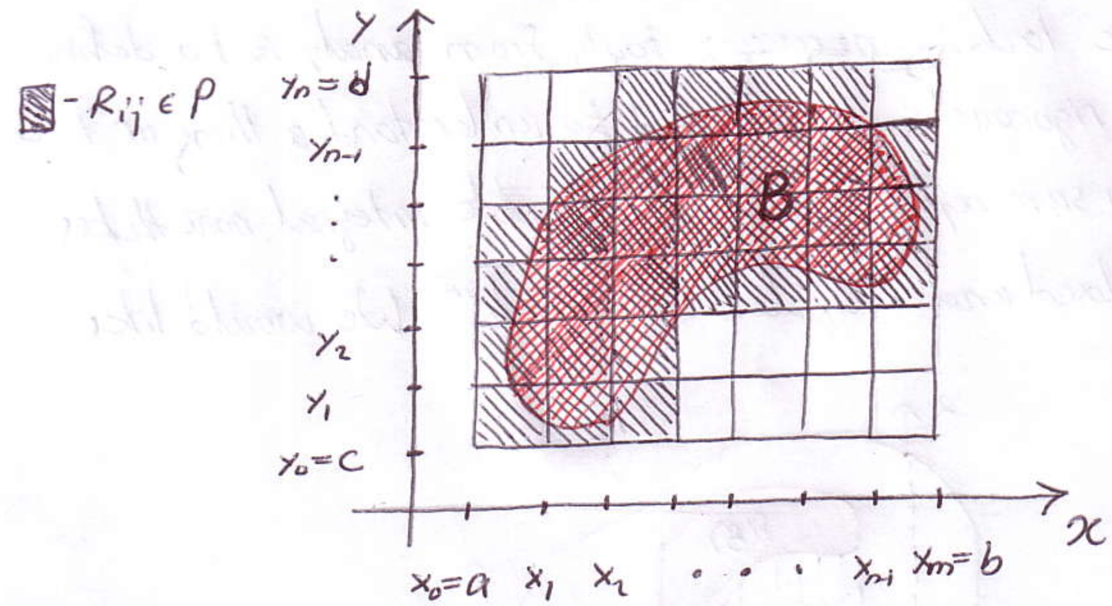
The assumption that B is bounded allows us to enclose B within some rectangle $R = [a,b] \times [c,d]$ as seen in the picture above.

We can partition the rectangle R into mn sub-rectangles, by partitioning the interval $[a,b]$ into m subintervals $a = x_0 < x_1 < x_2 < \ldots < x_m = b$ and by partitioning the interval $[c,d]$ $c = y_0 < y_1 < y_2 < \ldots < y_n = d$.

We thus have mn rectangles $R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$, $i \in \{1, \ldots, m\}$, $j \in \{1, \ldots, n\}$

Let P represent the collection of subrectangles R_{ij} that have non-empty intersection with B.

That is, $P = \{R_{ij} : R_{ij} \cap B \neq \emptyset\}$.



Observe that each $R_{ij} \in P$ is closed and bounded. Because B is also closed and bounded, it follows that $R_{ij} \cap B$ is a closed and bounded set. If f is continuous on B, f is also continuous on $R_{ij} \cap B$ for any $R_{ij} \in P$. By the extreme-value theorem, f attains both max and min on $R_{ij} \cap B$

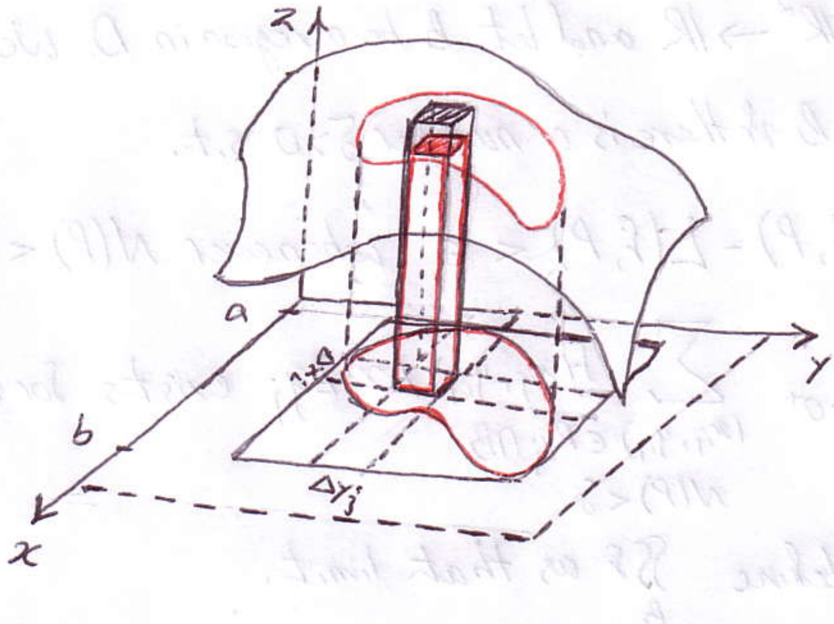
Define $U(f, P) = \sum_{(i,j):\, R_{ij} \in P} \max_{(x_j, y_j) \in R_{ij} \cap B} \{f(x_j, y_j)\} A(R_{ij}) =$

$$= \sum_{(i,j):\, R_{ij} \in P} \max_{(x,y) \in R_{ij} \cap B} f(x,y)\, \Delta x_i \Delta y_j$$

Similarly, define $L(f, P) = \sum_{(i,j):\, R_{ij} \in P} \min_{(x,y) \in R_{ij} \cap B} \{f(x,y)\} A(R_{ij}) =$

$$= \sum_{(i,j):\, R_{ij} \in P} \min_{(x,y) \in R_{ij} \cap B} f(x,y)\, \Delta x_i \Delta y_j$$

Intuitively speaking, $\mathcal{U}(f, P)$ is the biggest over-estimation of the volume above region B and below f over the partition P. On the other hand, $L(f, P)$ is the smallest under-estimation of the volume above region B and below f over the partition P.



The figure above shows a typical rectangle R_{ij}. The red box over R_{ij} has volume $\min\limits_{(x,y)\in R_{ij}\cap B} f(x,y)\,\Delta x_i \Delta y_j$ while the bigger black box has volume $\max\limits_{(x,y)\in R_{ij}\cap B} f(x,y)\,\Delta x_i \Delta y_j$.

Observe that for an arbitrary choice of points $(x_{ij}^*, y_{ij}^*) \in R_{ij}\cap B$

$$L(f,P) \leq \sum_{(x_{ij}^*,\, y_{ij}^*)\in R_{ij}\cap B} f(x_{ij}^*, y_{ij}^*)\,\Delta x_i \Delta y_j \leq \mathcal{U}(f,P)$$

Consider what may happen when we calculate Riemann sums $L(f,P)$ and $\mathcal{U}(f,P)$ for successively finer partitions P: the lower sums $L(f,P)$ will increase, while the upper sums $\mathcal{U}(f,P)$ will decrease. If both $L(f,P)$ and $\mathcal{U}(f,P)$ approach the same number, we may define $\iint_B f$ as that number.

To make this idea precise, it will be helpful to introduce a bit of terminology and notation.

Def: The norm of a partition P is the length of the longest diagonal of the rectangles in the partition. We denote the norm of P by $N(P)$.

Def: Let $f: D \subset \mathbb{R}^2 \to \mathbb{R}$ and let B be a region in D. We say that f is integrable over B if there is a number $\delta > 0$ s.t.

$$\mathcal{U}(f, P) - \mathcal{L}(f, P) < \epsilon \quad \text{whenever } N(P) < \delta.$$

In that case, $\lim\limits_{\delta \to 0^+} \sum\limits_{\substack{(x_{ij}, y_{ij}) \in R_{ij} \cap B \\ N(P) < \delta}} f(x_{ij}, y_{ij}) \Delta x_i \Delta y_j$ exists for any choice of (x_{ij}, y_{ij}). We define $\iint\limits_B f$ as that limit.

A natural question is, "which functions are integrable?" The following theorem classifies continuous functions over closed and bounded sets as integrable. This theorem is, unfortunately, beyond the scope of this text to prove.

Thm: If $f: D \subset \mathbb{R}^2 \to \mathbb{R}$ is continuous and B is a closed and bounded region contained in D, then f is integrable on B.

We have already seen that that, in the case when B is a rectangle of the form $[a,b] \times [c,d]$ and f is continuous on B, Cavalieri's principle may be applied to compute $\iint\limits_B f$. If f is not continuous, however, Cavalieri's principle does not always hold.

Ex. Find $\iint\limits_B \frac{x^2 - y^2}{(x^2+y^2)^2}$ where $B = [0,1] \times [0,1]$

Solution: Observe that $\frac{x^2-y^2}{(x^2+y^2)^2} = \frac{r^2\cos 2\theta}{r^4} = \frac{\cos 2\theta}{r^2}$

Because $\lim_{r\to 0^+} \frac{\cos 2\theta}{r^2}$ does not exist $\iint_B \frac{x^2-y^2}{(x^2+y^2)^2}$ is an improper integral. It can be shown that this integral does not exist.

Notice, however, that $\int_0^1\int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2}\,dx\,dy = -\frac{\pi}{4}$.

To see this, observe that $\int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2}\,dx = \int_0^{\tan^{-1}(\frac{1}{y})} \frac{y^2(\tan^2\theta-1)}{(y^2\sec^2\theta)^2}\,y\sec^2\theta\,d\theta$

$= \int_0^{\tan^{-1}(\frac{1}{y})} \frac{1}{y}(\sin^2\theta - \cos^2\theta)\,d\theta = -\frac{1}{y}\int_0^{\tan^{-1}(\frac{1}{y})} \cos 2\theta\,d\theta =$

$= -\frac{1}{y}\,\frac{\sin 2\theta}{2}\Big|_0^{\tan^{-1}(\frac{1}{y})} = -\frac{xy}{y(x^2+y^2)^2}\Big|_0^1 = -\frac{1}{1+y^2}$

where we used the substitution $x = y\tan\theta$.

$\int_0^1 \frac{-1}{1+y^2}\,dy = -\tan^{-1}(y)\Big|_0^1 = -\frac{\pi}{4}$.

By observing that $f(x,y) = \frac{x^2-y^2}{(x^2+y^2)^2}$ is skew symmetric, that is, by observing that $f(y,x) = -f(x,y)$, it follows that

$$\int_0^1\int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2}\,dy\,dx = -\int_0^1\int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2}\,dx\,dy = -\left(-\frac{\pi}{4}\right) = \frac{\pi}{4}$$

Thus, Cavalieri's principle does not work in this case

There are several properties of double integrals that are used frequently. We summarize them in the following theorem.

Thm: 1. If f is integrable on B and c is a constant, then

$$\iint_B cf = c\iint_B f$$

2. If f and g are integrable on B, then

$$\iint_B (f+g) = \iint_B f + \iint_B g$$

3. If f is integrable on $B = B_1 \cup B_2$ and $B_1 \cap B_2 = \emptyset$, then

$$\iint_B f = \iint_{B_1} f + \iint_{B_2} f.$$

4. If f and g are integrable on B and $g(x,y) \leq f(x,y)$ for all $(x,y) \in B$, then

$$\iint_B g \leq \iint_B f$$

5. (Mean Value theorem for double integrals). If f is continuous on a region B with area $A(B)$, then there exists a point $(a_1, a_2) \in B$ s.t.

$$\iint_B f = A(B)\, f(a_1, a_2).$$

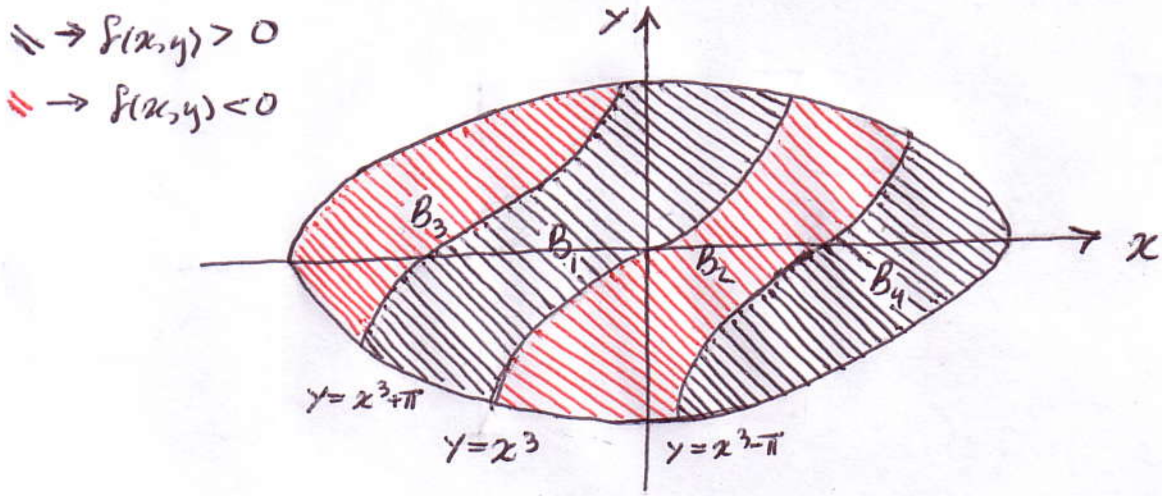
6. If f is continuous and satisfies $f(x,y) > 0$ for all $(x,y) \in B$, where B has nonempty interior, then

$$\iint_B f > 0$$

Ex. Evaluate $\iint_{\mathcal{E}} e^{x^2}\sin(y-x^3)$, where $\mathcal{E}$ is the ellipse $\{(x,y): \frac{x^2}{4}+y^2=1\}$.

Solution: Let $f(x,y)=e^{x^2}\sin(y-x^3)$. Observe that $f(-x,-y)=$ $=-f(x,y)$. Therefore, f is symmetric about the origin.

Notice that $f(x,y)>0$ when $y-x^3\in(0,\pi)$. In other words, whenever $x^3<y<x^3+\pi$. Similarly, $f(x,y)<0$ when $y-x^3\in(-\pi,0)$, or whenever $x^3-\pi<y<x^3$.



$$\iint_{\mathcal{E}} f = \iint_{B_1} f + \iint_{B_2} f + \iint_{B_3} f + \iint_{B_4} f.$$ Notice that $\iint_{B_2} f = -\iint_{B_1} f$

and $\iint_{B_3} f = -\iint_{B_4} f$ so $\iint_{\mathcal{E}} f = 0$.